

VOLUME AND NON-EXISTENCE OF COMPACT CLIFFORD–KLEIN FORMS

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ABSTRACT. This article studies the volume of compact quotients of reductive homogeneous spaces. Let G/H be a reductive homogeneous space and Γ a discrete subgroup of G acting properly discontinuously and cocompactly on G/H . We prove that the volume of $\Gamma \backslash G/H$ is the integral, over a certain homology class of Γ , of a G -invariant form on G/K (where K is a maximal compact subgroup of G).

As a corollary, we obtain a large class of homogeneous spaces the compact quotients of which have rational volume. For instance, compact quotients of pseudo-Riemannian spaces of constant curvature -1 and odd dimension have rational volume. This contrasts with the Riemannian case.

We also derive a new obstruction to the existence of compact Clifford–Klein forms for certain homogeneous spaces. In particular, we obtain that $\mathrm{SO}(p, q+1)/\mathrm{SO}(p, q)$ does not admit compact quotients when p is odd, and that $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(m, \mathbb{R})$ does not admit compact quotients when m is even.

INTRODUCTION

The problem of understanding compact quotients of homogeneous spaces has a long history that can be traced back to the “Erlangen program” of Felix Klein [15]. In the second half of the last century, the breakthroughs of Borel [6], Mostow [33], Margulis [29] and many others lead to a rather good understanding of quotients of *Riemannian* homogeneous spaces. Comparatively, little is known about the non-Riemannian case, and in particular about quotients of pseudo-Riemannian homogeneous spaces.

In this paper we will mainly focus on reductive homogeneous spaces, i.e. quotients of a semi-simple Lie group G by a closed reductive subgroup H . The G -homogeneous space $X = G/H$ carries a natural G -invariant pseudo-Riemannian metric (induced by the Killing metric of G) and therefore (up to taking a covering of degree 2) a G -invariant volume form vol_X . A quotient of X by a discrete subgroup Γ of G acting properly discontinuously and cocompactly is called a *compact Clifford–Klein form* of X , or (when it does not lead to any confusion) a *compact quotient* of X .

The study of compact reductive Clifford–Klein forms was initiated in the 80’s by Kulkarni [23] and Kobayashi [16]. A lot of things remain to be understood, despite the significant works of Benoist [3], Kobayashi [16, 17, 18, 19, 20], Labourie [4], Mozes and Zimmer [26], Margulis [28], and more recently the works of Kassel [12, 14], Guéritaud [11], Guichard and Wienhard [10].

In this paper we will address the following two questions, to which no general answer is known:

Existence Problem. *Which reductive homogeneous spaces admit compact Clifford–Klein forms?*

Volume Problem. *Let G/H be a reductive homogeneous space and Γ a discrete subgroup of G acting properly discontinuously and cocompactly on G/H . Is the volume of $\Gamma \backslash G/H$ rational (up to a scaling constant independent of Γ)?*

A particularly interesting family of homogeneous spaces are the *pseudo-Riemannian homogeneous spaces of constant curvature*, a unified definition of which was given by Wolf in [40]. Recall that the pseudo-Riemannian homogeneous space of signature (p, q) and constant negative curvature is the space

$$\mathbb{H}^{p,q} = \mathrm{SO}_0(p, q+1)/\mathrm{SO}_0(p, q) .$$

In this setting our results are summarized in the following:

THEOREM 1.

Let p and q be positive integers. Then:

- *If p is odd, then $\mathbb{H}^{p,q}$ does not admit any compact Clifford–Klein form.*
- *If p is even, then the volume of any compact Clifford–Klein form of $\mathbb{H}^{p,q}$ is a rational multiple of the volume of the sphere of dimension $p+q$.*

Prior to this work, the first point was only known when both p and q are odd [23], as well as when $p \leq q$ [40]. The second point follows from the Chern–Gauss–Bonnet formula when $p+q$ is even but is new when p is even and q is odd.

Let us now give a more detailed overview of the results contained in this paper.

Volume of Compact Clifford–Klein forms. It is well-known that the volume of a closed hyperbolic manifold of dimension $2n$ is essentially an integer, due to the Chern–Gauss–Bonnet formula. This argument generalizes to compact quotients of a reductive homogeneous space G/H whenever one can show that the volume is a *Chern–Weil class* associated to the canonical principal H -bundle over G/H (see Section 5). If G/H is a symmetric space, this is known to happen if and only if G and H have the same complex rank.

This argument has no chance to work for homogeneous spaces of odd dimension (because Chern–Weil classes have even degree), nor for homogeneous spaces of the form $H \times H/\Delta(H)$ (where $\Delta(H)$ denotes the diagonal embedding of H), for which all the Chern–Weil invariants are trivial. It is known for instance that the volume of a closed hyperbolic 3-manifold is usually not rational.

In contrast, we proved in a recent paper (see [38]) that the volume of a closed *anti-de Sitter* 3-manifold (i.e. a compact quotient of $\mathbb{H}^{2,1}$) is a rational multiple of $\frac{\pi^2}{2}$, answering a question that was raised in [2]. The anti-de Sitter space $\mathbb{H}^{2,1}$ can be seen as the *group space* $\mathrm{SO}_0(2, 1)$ (i.e. the

Lie group $\mathrm{SO}_0(2, 1)$ with the action of $\mathrm{SO}_0(2, 1) \times \mathrm{SO}_0(2, 1)$ by left and right multiplication, see Definition 6.1). Its compact Clifford–Klein are known to exist and to have a rich deformation space (see [34], [13] or [37]). Kulkarni and Raymond proved in [24] that these compact Clifford–Klein forms have the form

$$j \times \rho(\Gamma) \backslash \mathrm{SO}_0(2, 1) ,$$

where Γ is a cocompact lattice in $\mathrm{SO}_0(2, 1)$, j the inclusion and ρ another representation of Γ into $\mathrm{SO}_0(2, 1)$. Moreover, Guéritaud and Kassel proved in [11] that these quotients have the structure of a $\mathrm{SO}(2)$ -bundle over $\Gamma \backslash \mathbb{H}^2$ (see Theorem 8.1). In [38], we proved the following formula:

$$(1) \quad \mathbf{Vol}(j \times \rho(\Gamma) \backslash \mathrm{SO}_0(2, 1)) = \frac{\pi^2}{2} (\mathbf{eu}(j) + \mathbf{eu}(\rho)) ,$$

where \mathbf{eu} denotes the Euler class. This formula was later recovered by Alessandrini–Li [1] and Labourie [25] using different methods.

It may seem surprising that a “Chern–Weil-like” invariant such as the Euler class appears when computing the volume of a 3-manifold. The first aim of this paper is to explain better this phenomenon and generalize it to a much broader setting.

The main issue is that we don’t have a structure theorem similar to the one of Guéritaud–Kassel in general (see Theorem 8.1 and the conjecture that follows). We will overcome this problem with the following argument: denoting L a maximal compact subgroup of H and K a maximal compact subgroup of G containing L , we see that $\Gamma \backslash G/H$ is homotopically equivalent to $\Gamma \backslash G/L$, which is a K/L -bundle over $\Gamma \backslash G/K$. Let q be the dimension of K/L and $p + q$ the dimension of G/H . A classical use of spectral sequences shows that Γ has homological dimension p and that $H_p(\Gamma, \mathbb{Z})$ is generated by an element $[\Gamma]$ (Proposition 1.1). Since G/K is contractible, $H_p(\Gamma, \mathbb{Z})$ is naturally isomorphic to $H_p(\Gamma \backslash G/K, \mathbb{Z})$ and $[\Gamma]$ can thus be realized as a singular p -cycle in $\Gamma \backslash G/K$. We will prove the following:

THEOREM 2.

Let G/H be a reductive homogeneous space, with G and H connected and of finite center. Let L be a maximal compact subgroup of H and K a maximal compact subgroup of G containing L . Set $p = \dim G/H - \dim K/L$. Then there exists a G -invariant p -form $\omega_{G,H}$ on G/K such that, for any torsion-free discrete subgroup $\Gamma \subset G$ acting properly discontinuously and cocompactly on G/H , we have

$$\mathbf{Vol}(\Gamma \backslash G/H) = \left| \int_{[\Gamma]} \omega_{G,H} \right| .$$

It turns out that, in many cases, the form $\omega_{G,H}$ is a “Chern–Weil form”, though the volume form of G/H is not (see Section 4). This implies that the volume of any compact quotient of G/H is a rational multiple of the volume of G_U/H_U , where G_U and H_U respectively denote the compact Lie groups dual to G and H (see Section 5). In particular, we will obtain the following:

THEOREM 3.

For the following pairs (G, H) , the volume of compact quotients of G/H is a rational multiple of the volume of G_U/H_U :

- (1) $G = \mathrm{SO}(p, q + 1)$, $H = \mathrm{SO}(p, q)$, p even, $q > 0$.
- (2) $G = \mathrm{SL}(2n, \mathbb{R})$, $H = \mathrm{SL}(2n - 1, \mathbb{R})$, $n > 0$.
- (3) G a Hermitian Lie group, H any semi-simple subgroup.

Cases (1) and (2) concern families of symmetric spaces that have attracted a lot of interest. However, they potentially carry no information. Indeed the symmetric space $\mathrm{SL}(2n, \mathbb{R})/\mathrm{SL}(2n - 1, \mathbb{R})$ is conjectured not to admit any compact quotient (see next subsection), and the only known compact quotients of $\mathbb{H}^{p,q} = \mathrm{SO}(p, q + 1)/\mathrm{SO}(p, q)$ for $p \geq 3$ are the so-called *standard* quotients constructed by Kulkarni in [23], for which the theorem reduces to a classical statement about volumes of quotients of Riemannian symmetric spaces. Non standard quotients are only known in the case of $\mathbb{H}^{2,1}$, which was treated in [38] (see Equation (1)) and [1].

Case (3), on the other side, shows in particular that the volume of a compact quotient of the group space $\mathrm{SU}(d, 1)$ is a rational multiple of the volume of $\mathrm{SU}(d + 1)$. These compact quotients are known to exist and some of them have rich deformation spaces, as was proven by Kobayashi [], Kassel [14], and Guéritaud–Guichard–Kassel–Wienhard [10]. Like quotients of $\mathrm{SO}_0(2, 1)$, they are known to have (up to a finite cover) the form

$$j \times \rho(\Gamma) \backslash \mathrm{SU}(d, 1) ,$$

where Γ is a uniform lattice in $\mathrm{SU}(d, 1)$, $j : \Gamma \rightarrow \mathrm{SU}(d, 1)$ is the inclusion and $\rho : \Gamma \rightarrow \mathrm{SU}(d, 1)$ is another representation (see Theorem 6.2 for a more precise statement). For such Clifford–Klein forms, we will actually give a more precise formula. Recall that $\mathrm{SU}(d, 1)$ acts transitively on the complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^d$ and preserves a Kähler form ω . If Γ is a uniform lattice in $\mathrm{SU}(d, 1)$ and $\rho : \Gamma \rightarrow \mathrm{SU}(d, 1)$ a representation, we define

$$\tau_k(\rho) = \int_{\Gamma \backslash \mathbb{H}_{\mathbb{C}}^d} \omega^{d-k} \wedge f^* \omega^k ,$$

where $f : \mathbb{H}_{\mathbb{C}}^d \rightarrow \mathbb{H}_{\mathbb{C}}^d$ is any smooth ρ -equivariant map.

THEOREM 4.

Let Γ be a lattice in $\mathrm{SU}(d, 1)$, $j : \Gamma \rightarrow \mathrm{SU}(d, 1)$ the inclusion and $\rho : \Gamma \rightarrow \mathrm{SU}(d, 1)$ another representation such that $j \times \rho(\Gamma)$ acts properly discontinuously and cocompactly on $\mathrm{SU}(d, 1)$. Then

$$\mathrm{Vol}(j \times \rho(\Gamma) \backslash \mathrm{SU}(d + 1)) = \mathrm{Vol}(\mathrm{SU}(d + 1)) \sum_{k=0}^d \tau_k(\rho) .$$

A new obstruction to the existence of compact quotients. Contrary to the Riemannian setting, compact pseudo-Riemannian Clifford–Klein forms do not always exist, and it is a long standing problem to characterize which reductive homogeneous spaces admit compact quotients. This question lead to many important works of Kulkarni [23], Kobayashi [16, 17, 19], Benoist [3], Labourie, [4], Mozes, Zimmer [26, 27], Margulis [28] or Shalom [35]. We refer to [22] or [8] for a more thorough survey. Let us recall here two famous conjectures that emerged from these works.

Kobayashi’s Space-form Conjecture. *The homogeneous space $\mathbb{H}^{p,q} = \mathrm{SO}_0(p, q + 1)/\mathrm{SO}_0(p, q)$ ($p, q > 0$) admits a compact Clifford–Klein form if and only if one of the following holds:*

- p is even and $q = 1$,
- p is a multiple of 4 and $q = 3$,
- $p = 8$ and $q = 7$.

Conjecture (See for instance [13], Section 0.1.5). *The homogeneous space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(m, \mathbb{R})$ ($1 < m < n$) never admits a compact Clifford–Klein form.*

In this paper, we obtain a powerful cohomological obstruction, allowing us to do significant advances toward these conjectures. In Section 7, we prove that in many cases the form $\omega_{G,H}$ of Theorem 2 vanishes, directly implying that the reductive homogeneous space G/H does not admit a compact Clifford–Klein form. In particular, we obtain the following:

THEOREM 5.

For the following pairs (G, H) , the homogeneous space G/H does not have any compact Clifford–Klein form.

- (1) $G = \mathrm{SO}_0(p, q + r)$, $H = \mathrm{SO}_0(p, q)$, $p, q, r > 0$, p odd;
- (2) $G = \mathrm{SL}(n, \mathbb{R})$, $H = \mathrm{SL}(m, \mathbb{R})$, $1 < m < n$, m even;
- (4) $G = \mathrm{SL}(p + q, \mathbb{C})$, $H = \mathrm{SU}(p, q)$, $p, q > 0$;
- (5) $G = \mathrm{Sp}(2(p + q), \mathbb{C})$, $H = \mathrm{Sp}(p, q)$;
- (6) $G = \mathrm{SO}(2n, \mathbb{C})$, $H = \mathrm{SO}^*(2n)$;
- (7) $G = \mathrm{SL}(p + q, \mathbb{R})$, $H = \mathrm{SO}_0(p, q)$, $p, q > 1$;
- (8) $G = \mathrm{SL}(p + q, \mathcal{H})$, $H = \mathrm{Sp}(p, q)$, $p, q > 1$. (Here \mathcal{H} denotes the field of quaternions.)

All of these cases are partly new. They were obtained independently by Morita in [31]. We give more details about how these results relate to earlier works in Section 7.1 and to Yosuke Morita’s work in Section 7.2.

Finally, our obstruction will allow us to prove the following theorem, which was conjectured by Kobayashi (see [19, Conjecture 4.15]):

THEOREM 6.

Let G be a connected semi-simple Lie group, H a connected semi-simple subgroup of G , L a maximal compact subgroup of H and K a maximal compact subgroup of G containing L . If

$$\mathbf{rk}(G) - \mathbf{rk}(K) < \mathbf{rk}(H) - \mathbf{rk}(L)$$

(where \mathbf{rk} denotes the complex rank), then G/H does not have a compact Clifford–Klein form.

Note that Morita [30] independently proved that this theorem is implied by a previous result of his [32].

Organization of the paper. In Section 1, we explain why compact reductive Clifford–Klein forms behave like fibrations over an Eilenberg–MacLane space “at the homology level”. In Section 2 we construct the form $\omega_{G,H}$ as the contraction of a $(p + q)$ -form on G/L along the fibers gK/L and we prove Theorem 2. In Section 3, we study the form corresponding to $\omega_{G,H}$ on the compact dual symmetric space G_U/K and show that this form is “Poincaré-dual” to the inclusion of H_U/L in G_U/K . In Section 4, we derive a condition under which the form $\omega_{G,H}$ vanishes and a condition under

which it is a “Chern–Weil” class. In Section 5, we explain why, when $\omega_{G,H}$ is a Chern–Weil class, the volume of compact Clifford–Klein forms is rational, concluding the proof of Theorem 3. In Section 6, we describe the form $\omega_{G,H}$ in the case of group spaces and deduce Theorem 4. In Section 7 we give three different ways of proving the vanishing of the form $\omega_{G,H}$, leading to Theorems 5 and 6. Finally in Section 8, we prove that the vanishing of the form $\omega_{G,H}$ is also an obstruction to the existence of certain local foliations of G/H by compact homogeneous subspaces, and we formulate a conjecture about the geometry of compact reductive Clifford–Klein forms.

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1. CLIFFORD–KLEIN FORMS ARE FIBRATIONS AT THE HOMOLOGY LEVEL

In all this paper, G will denote a connected Lie group and H a closed connected subgroup of G . We will also fix L a maximal compact subgroup of H and K a maximal compact subgroup of G containing L . According to the Cartan–Iwasawa–Malcev theorem, L and K are well-defined up to conjugation. We denote respectively by \mathfrak{g} , \mathfrak{h} , \mathfrak{k} and \mathfrak{l} the Lie algebras of G , H , K and L .

We will assume that the action of G on the homogeneous space $X = G/H$ preserves a volume form. Recall that this is equivalent to requiring that

$$\mathrm{Det}(G)|_H = \mathrm{Det}(H) ,$$

where $\mathrm{Det}(G)$ and $\mathrm{Det}(H)$ denote respectively the *modular functions* of G and H . Starting from Section 3, we will assume G and H to be reductive and therefore unimodular, in which case this condition is automatically satisfied.

A *compact Clifford–Klein form* of X is a quotient of X by a discrete subgroup Γ of G acting properly discontinuously and cocompactly. The G -invariant volume form vol_X then descends to a volume form on $\Gamma \backslash X$ (that we still denote by vol_X) and we can define the *volume* of $\Gamma \backslash X$ by

$$\mathbf{Vol}(\Gamma \backslash X) = \left| \int_{\Gamma \backslash X} \mathrm{vol}_X \right| .$$

Recall that, since K and L are maximal compact subgroups of G and H respectively, the homogeneous spaces G/K and H/L are contractible. Let us fix a torsion-free discrete subgroup Γ of G acting properly discontinuously and cocompactly on G/H , and denote by M the Clifford–Klein form

$$M = \Gamma \backslash G/H .$$

We introduce two auxiliary Clifford–Klein forms:

$$E = \Gamma \backslash G/L$$

and

$$B = \Gamma \backslash G/K .$$

(E and B are smooth manifolds since Γ is discrete and torsion-free.)

We remark the following facts:

- (i) E fibers over M with fibers isomorphic to H/L . Since H/L is contractible, this fibration is a homotopy equivalence.
- (ii) E also fibers over B with fibers isomorphic to K/L .
- (iii) Since G/K is contractible, B is a *classifying space* for Γ .

From the first point, we deduce in particular that the homology of M is the same as the homology of E . The third point implies that the homology of B is the homology of Γ . Finally, (ii) implies that the homologies of B , E and K/L are linked (in an elaborate way) by the *Leray–Serre spectral sequence*. We will use the following classical consequence:

Proposition 1.1 (See [23] and [16]). *Let q denote the dimension of K/L and $p + q$ the dimension of G/H . Then the group Γ has homological dimension p and*

$$H_p(\Gamma, \mathbb{Z}) \simeq H_{p+q}(M, \mathbb{Z}) \simeq \mathbb{Z} .$$

Proof. Let p' , q' and r' denote respectively the homological dimensions of B , K/L and E . By Serre's theorem, the spectral sequence given by

$$\mathbf{E}_{k,l}^2 = H_k(B, H_l(K/L, \mathbb{Z}))$$

converges to $H_{k+l}(E, \mathbb{Z})$. A classical consequence is that

$$r' = p' + q'$$

and that

$$(2) \quad H_{p'+q'}(E, \mathbb{Z}) \simeq H_{p'}(B, H_{q'}(K/L, \mathbb{Z})) .$$

Since K/L is a closed oriented manifold of dimension q , we have $q' = q$ and $H_q(K/L, \mathbb{Z}) \simeq \mathbb{Z}$. Since E is homotopy equivalent to M which is a closed oriented manifold of dimension $p + q$, we also have $r' = p + q$. Therefore $p' = p$.

Moreover, since L is connected, the action of Γ on G/L preserves an orientation of the fibers of the fibration

$$G/L \rightarrow G/K$$

and Γ thus acts trivially on $H_q(K/L, \mathbb{Z})$. From (2), we obtain

$$\mathbb{Z} \simeq H_{p+q}(E, \mathbb{Z}) \simeq H_p(B, \mathbb{Z}) .$$

The proposition follows since E is homotopy equivalent to M and B is a classifying space for Γ . \square

To go further, we need to explicitly describe the isomorphism $H_{p+q}(E, \mathbb{Z}) \simeq H_p(B, \mathbb{Z})$. Let $[\Gamma]$ denote a generator of $H_p(B, \mathbb{Z}) \simeq H_p(\Gamma, \mathbb{Z})$, and π the fibration of E over B . Roughly speaking, if one thinks of $[\Gamma]$ as a closed submanifold of B of dimension p , then the isomorphism $H_p(B, \mathbb{Z}) \rightarrow H_{p+q}(E, \mathbb{Z})$ maps $[\Gamma]$ to $\pi^{-1}([\Gamma])$, which is a submanifold of E of dimension $p + q$.

However, we don't know whether $[\Gamma]$ can be represented by a submanifold. One way to overcome this difficulty would be to work with simplicial complexes. However, since we will use differential geometry later, it is more convenient to use Thom's realization theorem:

Theorem 1.2 (Thom, [39]). *There exists a closed oriented p -manifold B' , a smooth map $\varphi : B' \rightarrow B$ and an integer k such that*

$$k[\Gamma] = \varphi_*[B'] ,$$

where $[B']$ denotes the fundamental class of B' .

Let $\pi' : E' \rightarrow B'$ be the pull-back of the fibration $\pi : E \rightarrow B$ by φ and $\hat{\varphi} : E' \rightarrow E$ the lift of φ . The total space of the fibration E' is a closed orientable $(p+q)$ -manifold.

Proposition 1.3. *Let $[E]$ denote a generator of $H_{p+q}(E)$ and $[E']$ denote the fundamental class of E' . Then, up to switching the orientation of E' , we have*

$$k[E] = \hat{\varphi}_*[E'] .$$

Proof. The Leray–Serre spectral sequence shows that the fibrations π and π' respectively induce isomorphisms

$$\pi^* : H_p(B) \rightarrow H_{p+q}(E)$$

and

$$\pi'^* : H_p(B') \rightarrow H_{p+q}(E') .$$

By naturality of the Serre spectral sequence, we have the following commuting diagram:

$$\begin{array}{ccc} H_p(B') & \xrightarrow{\varphi_*} & H_p(B) \\ \pi'^* \downarrow & & \downarrow \pi^* \\ H_{p+q}(E') & \xrightarrow{\hat{\varphi}_*} & H_{p+q}(E) . \end{array}$$

Now, B' and E' are closed oriented manifolds of dimension p and $p+q$ respectively. Since π'^* is an isomorphism, it maps the fundamental class of B' to the fundamental class of E' (up to switching the orientation of E'). Since $\varphi_*[B'] = k[\Gamma]$, we thus have

$$\hat{\varphi}_*[E'] = k[E] .$$

□

To summarize, we proved that the rational homology of E in dimension $p+q$ is generated by a cycle that “fibers” over a p -cycle of B .

2. FIBERWISE INTEGRATION OF THE VOLUME FORM

Let E' , B' , φ , $\hat{\varphi}$ and π , π' be as in the previous section. Denote by ψ the projection from E to M . Recall that the volume form vol_X on $X = G/H$ induces a volume form on M that we still denote by vol_X .

Since ψ is a homotopy equivalence, we have

$$\mathbf{Vol}(M) = \left| \int_M vol_X \right| = \left| \int_{[E]} \psi^* vol_X \right| .$$

Since $k[E] = \hat{\varphi}_*[E']$, we have

$$\left| \int_{[E]} \psi^* vol_X \right| = \frac{1}{k} \left| \int_{E'} \hat{\varphi}^* \psi^* vol_X \right| .$$

Now, since E' fibers over B' , we can “average” the form $\hat{\varphi}^*\psi^*vol_X$ along the fibers to obtain a p -form on B' whose integral will give the volume of M . Let x be a point in G/K and let F denote the fiber $\pi^{-1}(x)$. Choose some volume form vol_F on F and let ξ denote the section of $\Lambda^q TF$ such that $vol_F(\xi) = 1$. At every point y of F , the p -form obtained by contracting ψ^*vol_X with ξ has $T_y F$ in its kernel and therefore induces a p -form ω_y on $T_x G/K$.

Definition 2.1. The form $\omega_{G,H}$ on G/K is defined at the point x by

$$(\omega_{G,H})_x = \int_F \omega_y \, dvol_F(y) .$$

One easily checks that this definition does not depend on the choice of vol_F . Since the maps ψ and π are equivariant with respect to the actions of G , the volume forms ψ^*vol_X and $\omega_{G,H}$ are G -invariant. By a slight abuse of notation, we still denote by $\omega_{G,H}$ the induced p -form on $B = \Gamma \backslash G/K$.

Proposition 2.2. For any submanifold V of dimension p in G/K , we have

$$(3) \quad \int_V \omega_{G,H} = \int_{\pi^{-1}(V)} \psi^*vol_X .$$

Proof. This is presumably a classical result of differential geometry. Let U be an open subset of V over which the fibration π is trivial. Let us identify $\pi^{-1}(U)$ with $K/L \times U$. We can locally write the form ψ^*vol_X as $f(y,x)vol_F \wedge vol_U$ for some function f on $K/L \times U$ and some volume forms vol_F and vol_U on K/L and U respectively. Let ξ be the section of $\Lambda^q TK/L$ such that $vol_F(\xi) = 1$. The contraction of ψ^*vol_X with ξ is thus $f(y,x)vol_U$. By construction, we thus have

$$(\omega_{G,H})_x = \left(\int_F f(x,y) dvol_F(y) \right) vol_U ,$$

and therefore

$$\begin{aligned} \int_{\pi^{-1}(U)} \psi^*vol_X &= \int_{F \times U} f(y,x) dvol_F(y) dvol_U(x) \\ &= \int_U \omega_{G,H} . \end{aligned}$$

□

In particular, if V is a sphere of dimension p in G/K that can be homotoped to a point p , then $\pi^{-1}(V)$ can be homotoped to the fiber $\pi^{-1}(p)$. We thus have

$$\int_V \omega_{G,H} = \int_{\pi^{-1}(V)} \psi^*vol_X = 0 .$$

Since ψ^*vol_X is closed. This shows that $\omega_{G,H}$ is closed.

Remark 2.3. In the following, we will assume that G is semi-simple, in which case any G -invariant form on G/K is closed, according to a well-known theorem of Cartan.

We can now conclude the proof of Theorem 2. Indeed, we have

$$\begin{aligned}
\mathbf{Vol}(M) &= \frac{1}{k} \left| \int_{E'} \hat{\varphi}^* \psi^* \text{vol}_X \right| \\
&= \frac{1}{k} \left| \int_{B'} \varphi^* \omega_{G,H} \right| \quad \text{by Proposition (2.2)} \\
&= \left| \int_{[\Gamma]} \omega_{G,H} \right|.
\end{aligned}$$

Let us conclude this section by giving a more explicit way to compute the form $\omega_{G,H}$ when G is a connected semi-simple Lie group with finite center. Recall that in that case, the tangent space of G/K at the point $x_0 = K$ can be identified with the orthogonal of \mathfrak{h} in \mathfrak{g} with respect to the Killing form of \mathfrak{g} . Moreover, the form $\omega_{G,H}$ is uniquely determined by its restriction to $T_{x_0}G/K$.

If \mathfrak{v} is a subspace of \mathfrak{g} of dimension d in restriction to which the Killing form κ_G is non degenerate, we denote by $\omega_{\mathfrak{v}}$ the d -form on \mathfrak{g} given by composing the orthogonal projection on \mathfrak{v} with the volume form on \mathfrak{v} induced by the restriction of the Killing form.

Finally, let us provide K/L with the left invariant volume form $\omega_{K/L}$ induced by the restriction of the metric on G/H .

Lemma 2.4. *The form $\omega_{G,H}$ at the point x_0 is given by*

$$(\omega_{G,H})_{x_0} = \int_{K/L} \text{Ad}_u^* \omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp} \, d\omega_{K/L}(u) .$$

Proof. In the construction of $\omega_{G,H}$ (Definition 2.1), we choose $\omega_{K/L}$ as our volume form on $F_{x_0} = K/L$. Let ξ be the q -vector on $\omega_{K/L}$ such that $\omega_{K/L}(\xi) = 1$.

At $y_0 = L$, the pull-back of vol_X by the projection $\psi : G/L \rightarrow G/H$ identifies with the form $\omega_{\mathfrak{h}^\perp}$ on \mathfrak{g} . Since the q -vector ξ at y_0 is given by $e_1 \wedge \dots \wedge e_q$, where (e_1, \dots, e_q) is an orthonormal frame of $\mathfrak{k} \cap \mathfrak{h}^\perp$, we have

$$(i_\xi \omega_{\mathfrak{h}^\perp})_{y_0} = \omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp} .$$

By left invariance, we also have

$$(i_\xi \psi^* \text{vol}_X)_{u \cdot y_0} = u_* \omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp} .$$

Now, identifying $T_{u \cdot y_0} G/L$ with $u_* \mathfrak{l}^\perp$, the differential of $\pi : G/L \rightarrow G/K$ is given at $u \cdot y_0$ by

$$\begin{aligned}
d\pi_{u \cdot y_0}(u_* v) &= \frac{d}{dt} \pi(u \exp(tv) \cdot y_0) \\
&= \frac{d}{dt} \pi(\exp(t \text{Ad}_u(v)) u \cdot y_0) \\
&= \frac{d}{dt} \exp(t \text{Ad}_u(v)) \cdot \pi(u \cdot y_0) \\
&= \frac{d}{dt} \exp(t \text{Ad}_u(v)) \cdot x_0 \\
&= p_{\mathfrak{k}^\perp} \text{Ad}_u(v) ,
\end{aligned}$$

where $p_{\mathfrak{k}^\perp}$ denotes the orthogonal projection on \mathfrak{k}^\perp .

Therefore, the form $(i_\xi \psi^* \text{vol}_X)$ at $u \cdot y_0$, whose kernel contains $u_* \mathfrak{k}$, induces by projection the form $\text{Ad}_{u*} \omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp}$ at x_0 . By construction of the form $\omega_{G,H}$, we thus obtain

$$(\omega_{G,H})_{x_0} = \int_{K/L} \text{Ad}_{u*} \omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp} \, d\omega_{K/L}(u) .$$

□

3. THE CORRESPONDING FORM ON THE COMPACT DUAL

From now on, we assume that G is a connected semi-simple Lie group with finite center and that H is a reductive subgroup. In this section we investigate the form $\omega_{G,H}^U$ corresponding to $\omega_{G,H}$ on the *compact dual* of G/K .

Write

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} ,$$

where \mathfrak{p} is the orthogonal of \mathfrak{k} with respect to the Killing form. Then $\mathfrak{k} \oplus i\mathfrak{p}$ is a Lie subalgebra of the complexification $\mathfrak{g}^\mathbb{C}$ of \mathfrak{g} , generating a compact Lie group G_U containing K , called the *compact dual* of G . The compact symmetric space G_U/K is the *compact dual* of the symmetric space G/K .

By construction, the tangent spaces at the base point $x_0 = K$ in G/K and G_U/K are isomorphic as representations of K . This induces an isomorphism between the exterior algebras of invariant forms on G/K and G_U/K . If α is a G -invariant form on G/K , the image of α by this isomorphism will be called the *form corresponding to α on the compact dual* and will be denoted α^U .

The group G_U contains the compact dual H_U of H , and one can define a map $\iota : H_U/L \rightarrow G_U/K$. This map may not be injective, but it is a covering of finite degree onto its image, since L is a finite index subgroup of $H_U \cap K$. We denote by $[H_U/L]$ the fundamental class of H_U/L .

Definition 3.1. Let N be a closed oriented manifold of dimension d and $[c]$ a rational homology class of degree k on N . Let

$$\vee : H_k(N, \mathbb{Q}) \times H_{n-k}(N, \mathbb{Q}) \rightarrow \mathbb{Q}$$

denote the intersection pairing. The cohomology class $[\alpha] \in H^{d-k}(N, \mathbb{Q})$ is called *Poincaré-dual* to $[c]$ if for any $[c'] \in H_{d-k}(N, \mathbb{Q})$, one has

$$\int_{[c']} [\alpha] = [c] \vee [c'] .$$

According to Poincaré's duality theorem, every rational homology class of a closed oriented manifold has a unique Poincaré-dual cohomology class.

Theorem 3.2. *The cohomology class of the form*

$$\frac{1}{\text{Vol}(G_U/H_U)} \omega_{G,H}^U \in H^\bullet(G_U/K, \mathbb{Q})$$

is Poincaré-dual to the homology class $\iota_[H_U/L]$.*

Proof. Let x_0 denote the point K in G_U/K . By Lemma 2.4, we have

$$(\omega_{G,H}^U)_{x_0} = \int_{K/L} \text{Ad}_u^* \omega_{\mathfrak{k}^\perp \cap i\mathfrak{h}^\perp} \, d\omega_{K/L}(u) .$$

Thus, if φ denotes the projection from G_U/L to G_U/H_U and π the projection from G_U/L to G_U/K , then one can reproduce word by word the arguments of the previous section and show that

$$\int_C \frac{1}{\text{Vol}(G_U/H_U)} \omega_{G,H}^U = \int_{\pi^{-1}(C)} \frac{1}{\text{Vol}(G_U/H_U)} \varphi^* \text{vol}_{G_U/H_U}$$

for any oriented submanifold C of G_U/K of dimension p .

Now, the form $\frac{1}{\text{Vol}(G_U/H_U)} \text{vol}_{G_U/H_U}$ is Poincaré-dual to the homology class of a point in G_U/H_U , and $\varphi^* \text{vol}_{G_U/H_U}$ is thus dual to the homology class of the fiber $H_U/L \subset G_U/L$ of the map φ .

Therefore, $\int_{\pi^{-1}(C)} \frac{1}{\text{Vol}(G_U/H_U)} \varphi^* \text{vol}_{G_U/H_U}$ counts the homological intersection number between H_U/L and $\pi^{-1}(C)$ in G_U/L . This is equal to k times the homological intersection number between $\iota(H_U/L)$ and C in G_U/K , where k denotes the degree of the covering map $\iota : H_U/L \rightarrow H_U/H_U \cap K$. Hence $\int_{\pi^{-1}(C)} \frac{1}{\text{Vol}(G_U/H_U)} \varphi^* \text{vol}_{G_U/H_U}$ is equal to $[C] \vee \iota_*[H_U/L]$. The conclusion follows. \square

4. COHOMOLOGY AND INCLUSION OF SYMMETRIC SPACES

In this section, we go deeper into the cohomology theory of symmetric spaces in order to find conditions under which the form $\omega_{G,H}^U$ vanishes and conditions under which it is a *Chern–Weil form*.

We say that $\omega_{G,H}^U$ is a *Chern–Weil form* if its cohomology class is a Chern–Weil characteristic class of the canonical principal K -bundle over G_U/K (see Section 5 for details). Our aim is to prove the following theorem:

Theorem 4.1. *Let rk denote the complex rank of a Lie group.*

- *The form $\omega_{G,H}^U$ vanishes when*

$$\text{rk}(H_U) - \text{rk}(L) > \text{rk}(G_U) - \text{rk}(K) .$$

- *If $\omega_{G,H}^U$ does not vanish, then it is a Chern–Weil form if and only if*

$$\text{rk}(H_U) - \text{rk}(L) = \text{rk}(G_U) - \text{rk}(K) .$$

The cohomology of symmetric spaces has been described by the works of Cartan and Borel in the years 1950 [7, 5]. This description is summarized in the following theorem:

Theorem 4.2 (Cartan). *Let G_U/K be a symmetric space of compact type, with K connected. Then*

- *The cohomology algebra $H^\bullet(G_U/K, \mathbb{Q})$ is isomorphic to a tensor product*

$$H_{\text{even}}^\bullet(G_U/K, \mathbb{Q}) \otimes H_{\text{odd}}^\bullet(G_U/K, \mathbb{Q}) ,$$

- *the subalgebra $H_{\text{even}}^\bullet(G_U/K, \mathbb{Q})$ is the algebra of Chern–Weil classes of the canonical principal K -bundle over G_U/K , and is concentrated in even degree,*

- the subalgebra $H_{\text{odd}}^{\bullet}(G_U/K, \mathbb{Q})$ is isomorphic to $\Lambda^{\bullet}(\text{Prim}(G_U/K, \mathbb{Q}))$ where $\text{Prim}(G_U/K, \mathbb{Q})$ is a vector subspace of dimension $\text{rk}(G_U) - \text{rk}(K)$ generated by elements of odd degree,

The cohomology algebra of a symmetric space thus has the structure of a bi-graded algebra:

$$H^{\bullet}(G_U/K, \mathbb{Q}) = \bigoplus_{p,q \geq 0} H_{\text{even}}^p(G_U/K, \mathbb{Q}) \otimes H_{\text{odd}}^q(G_U/K, \mathbb{Q}) .$$

We will say that a cohomology class α has bi-degree (p, q) if it belongs to $H_{\text{even}}^p(G_U/K, \mathbb{Q}) \otimes H_{\text{odd}}^q(G_U/K, \mathbb{Q})$.

Proposition 4.3. *The map $\iota^* : H^{\bullet}(G_U/K, \mathbb{Q}) \rightarrow H^{\bullet}(H_U/L, \mathbb{Q})$ maps $H_{\text{even}}^p(G_U/K, \mathbb{Q})$ to $H_{\text{even}}^p(H_U/L, \mathbb{Q})$ and $H_{\text{odd}}^p(G_U/K, \mathbb{Q})$ to $H_{\text{odd}}^p(H_U/L, \mathbb{Q})$, and thus preserves the bi-grading. Moreover, it maps $\text{Prim}(G_U/K, \mathbb{Q})$ to $\text{Prim}(H_U/L, \mathbb{Q})$.*

This proposition is likely to be a straightforward consequence of the proof of Cartan's theorem. We prove it in the forthcoming paper [36].

If G_U/K is a symmetric space of compact type, let us denote by $d_e(G_U/K)$ and $d_o(G_U/K)$ the maximal degree of a non zero cohomology class in $H_{\text{even}}^{\bullet}(G_U/K, \mathbb{Q})$ and $H_{\text{odd}}^{\bullet}(G_U/K, \mathbb{Q})$, respectively. Since G_U/K is compact and orientable, we obtain by Cartan's theorem that

$$d_e(G_U/K) + d_o(G_U/K) = \dim(G_U/K)$$

and that

$$H_{\text{even}}^{d_e(G_U/K)}(G_U/K, \mathbb{Q}) \otimes H_{\text{odd}}^{d_o(G_U/K)}(G_U/K, \mathbb{Q}) = H^{\dim(G_U/K)}(G_U/K, \mathbb{Q}) .$$

Thus, both $H_{\text{even}}^{d_e(G_U/K)}(G_U/K, \mathbb{Q})$ and $H_{\text{odd}}^{d_o(G_U/K)}(G_U/K, \mathbb{Q})$ have dimension 1.

Proposition 4.4. *If $\iota_*[H_U/L]$ does not vanish in $H_{\bullet}(G_U/K, \mathbb{Q})$, then the homomorphism*

$$\iota^* : H_{\text{even}}^{d_e(H_U/L)}(G_U/K, \mathbb{Q}) \rightarrow H_{\text{even}}^{d_e(H_U/L)}(H_U/L, \mathbb{Q})$$

is surjective, and the morphism

$$\iota^* : \text{Prim}(G_U/K, \mathbb{Q}) \rightarrow \text{Prim}(H_U/L, \mathbb{Q})$$

is surjective.

Proof. If $\iota_*[H_U/L]$ does not vanish in $H_{\bullet}(G_U/K, \mathbb{Q})$, then, by Poincaré duality, there exists an element $\alpha \in H^{\dim(H_U/L)}(G_U/K, \mathbb{Q})$ such that $\iota^*\alpha \neq 0$. By Cartan's theorem, we can write

$$\alpha = \sum_{k+l=\dim(H_U/L)} \beta_k \otimes \gamma_l ,$$

with $\beta_k \in H_{\text{even}}^k(G_U/K, \mathbb{Q})$ and $\gamma_l \in H_{\text{odd}}^l(G_U/K, \mathbb{Q})$.

Since $\iota^*\beta_k = 0$ for $k > d_e(H_U/L)$ and $\iota^*\gamma_l = 0$ for $l > d_o(H_U/L)$, we get that

$$\iota^*\alpha = \iota^*\beta_{d_e(H_U/L)} \otimes \iota^*\gamma_{d_o(H_U/L)} \neq 0 ,$$

which implies that both $i^* \beta_{d_e(H_U/L)}$ and $i^* \gamma_{d_o(H_U/L)}$ do not vanish. Since $H^{d_e(H_U/L)}(H_U/L, \mathbb{Q})$ and $H^{d_o(H_U/L)}(H_U/L, \mathbb{Q})$ are one dimensional, we conclude that

$$\iota^* : H_{even}^{d_e(H_U/L)}(G_U/K, \mathbb{Q}) \rightarrow H_{even}^{d_e(H_U/L)}(H_U/L, \mathbb{Q})$$

and

$$\iota^* : H_{odd}^{d_o(H_U/L)}(G_U/K, \mathbb{Q}) \rightarrow H_{odd}^{d_o(H_U/L)}(H_U/L, \mathbb{Q})$$

are surjective.

Now, by Cartan's theorem, $H_{odd}^\bullet(H_U/L, \mathbb{Q}) = \Lambda^\bullet \text{Prim}(H_U/L, \mathbb{Q})$. If $\iota^* : \text{Prim}(G_U/K, \mathbb{Q}) \rightarrow \text{Prim}(H_U/L, \mathbb{Q})$ were not surjective, then $\iota^*(H_{odd}^\bullet(G_U/K, \mathbb{Q}))$ would be included in $\Lambda^\bullet F$ for a proper subspace F of $\text{Prim}(H_U/L, \mathbb{Q})$, and it would not contain any form of top degree. Since

$$\iota^* : H_{odd}^{d_o(H_U/L)}(G_U/K, \mathbb{Q}) \rightarrow H_{odd}^{d_o(H_U/L)}(H_U/L, \mathbb{Q})$$

is surjective, we conclude that $\iota^* : \text{Prim}(G_U/K, \mathbb{Q}) \rightarrow \text{Prim}(H_U/L, \mathbb{Q})$ is surjective. \square

We can now prove Theorem 4.1.

Proof of Theorem 4.1. Assume that $\omega_{G,H}^U$ does not vanish. Then, by Proposition 3.2, $\iota_*[H_U/L]$ does not vanish in $H_\bullet(G_U/K, \mathbb{Q})$. By Proposition 4.4, the map $\iota^* : \text{Prim}(G_U/K, \mathbb{Q}) \rightarrow \text{Prim}(H_U/L, \mathbb{Q})$ is surjective, which implies that

$$\mathbf{rk}(H_U) - \mathbf{rk}(L) = \dim \text{Prim}(H_U/L, \mathbb{Q}) \leq \dim \text{Prim}(G_U/K, \mathbb{Q}) = \mathbf{rk}(G_U) - \mathbf{rk}(K) .$$

This proves the first point.

Now, since $\omega_{G,H}^U$ is Poincaré dual to $i_*[H_U/L]$, we have

$$\int_{H_U/L} i^* \alpha = \int_{G_U/K} \alpha \wedge \omega_{G,H}^U$$

for all $\alpha \in H^{\dim(H_U/L)}(G_U/K, \mathbb{Q})$. In particular, for all (k, l) such that $k + l = \dim(H_U/L)$ and for all $\alpha \in H^{k+l}(G_U/K, \mathbb{Q})$ of bi-degree (k, l) , we have $\int_{G_U/K} \alpha \wedge \omega_{G,H}^U = 0$ unless

$$(k, l) = (d_e(H_U/L), d_o(H_U/L)) .$$

This implies that $\omega_{G,H}^U$ has bi-degree

$$(d_e(G_U/K) - d_e(H_U/L), d_o(G_U/K) - d_o(H_U/L)) .$$

Therefore, $[\omega_{G,H}^U]$ belongs to $H_{even}^\bullet(G_U/K, \mathbb{Q})$ if and only if

$$(4) \quad d_o(G_U/K) = d_o(H_U/L) .$$

Since $\iota^* : \text{Prim}(G_U/K, \mathbb{Q}) \rightarrow \text{Prim}(H_U/L, \mathbb{Q})$ is surjective, Equality (4) happens if and only if it is also injective, which is equivalent to

$$\mathbf{rk}(H_U) - \mathbf{rk}(L) = \mathbf{rk}(G_U) - \mathbf{rk}(K) .$$

This concludes the proof of Theorem 4.1. \square

5. CHARACTERISTIC CLASSES AND RATIONALITY OF THE VOLUME

In this section, we explain why, when $\omega_{G,H}^U$ is a Chern–Weil form, the volume of every compact quotient of G/H is a rational multiple of $\mathbf{Vol}(G_U/H_U)$. This is a classical argument which relies on the fact that, by Proposition 3.2, the form $\frac{1}{\mathbf{Vol}(G_U/H_U)}\omega_{G,H}^U$ represents an *integral* cohomology class.

The precise result that we will prove is the following:

Theorem 5.1. *Assume that we have the equality:*

$$\mathbf{rk}(G_U) - \mathbf{rk}(K) = \mathbf{rk}(H_U) - \mathbf{rk}(L) .$$

Then there exists an integer d such that, for any torsion-free discrete subgroup of G acting properly discontinuously on G/H , the volume $\mathbf{Vol}(\Gamma \backslash G/H)$ is an integral multiple of $\frac{1}{d}\mathbf{Vol}(G_U/H_U)$.

Remark 5.2. Note that, given a normalization of the volume form on G/H , there is a canonical way to normalize the volume on G_U/H_U accordingly. Thus the statement of Theorem 5.1 does not depend on the choice of such a normalization.

Proof of Theorem 5.1. Let BK be a classifying space for K and $EK \rightarrow BK$ be the associated universal principal K -bundle. There exists a map $f : G_U/K \rightarrow BK$, unique up to homotopy, such that the principal K -bundle G_U is isomorphic to f^*EK . The map f induces a homomorphism

$$f^* : H^\bullet(BK, \mathbb{R}) \rightarrow H^\bullet(G_U/K, \mathbb{R}) .$$

By Theorem 4.2 and by definition of Chern–Weil classes, the image of f^* is the subalgebra $H_{\text{even}}^\bullet(G_U/K, \mathbb{R})$. It contains as a lattice the \mathbb{Z} -module $f^*H^\bullet(BK, \mathbb{Z})$.

It follows from Proposition 3.2 that the form $\frac{1}{\mathbf{Vol}(G_U/H_U)}\omega_{G,H}^U$ represents an integral cohomology class. Moreover, we saw in the previous section that, under the condition $\mathbf{rk}(G_U) - \mathbf{rk}(K) = \mathbf{rk}(H_U) - \mathbf{rk}(L)$, this cohomology class belongs to $H_{\text{even}}^\bullet(G_U/K, \mathbb{R})$. Therefore, the cohomology class $\frac{1}{\mathbf{Vol}(G_U/H_U)}[\omega_{G,H}^U]$ belongs to the \mathbb{Z} -module $\Lambda = H_{\text{even}}^\bullet(G_U/K, \mathbb{R}) \cap H^\bullet(G_U/K, \mathbb{Z})$. Since we have

$$f^*H^\bullet(BK, \mathbb{Z}) \subset \Lambda$$

and since $f^*H^\bullet(BK, \mathbb{Z})$ is a lattice in $H_{\text{even}}^\bullet(G_U/K, \mathbb{R})$, we obtain that $f^*H^\bullet(BK, \mathbb{Z})$ has finite index in Λ . Therefore, there exists an integer d such that

$$\frac{d}{\mathbf{Vol}(G_U/H_U)}[\omega_{G,H}^U] \in f^*H^\bullet(BK, \mathbb{Z}) .$$

Let us now denote by $\text{Sym}^\bullet(\mathfrak{k})^K$ the algebra of polynomials on \mathfrak{k} invariant by the adjoint action of K . The Chern–Weil theory gives the existence of an isomorphism

$$\Phi : H^\bullet(BK, \mathbb{R}) \rightarrow \text{Sym}^\bullet(\mathfrak{k})^K$$

such that, for any smooth map f from a manifold M to BK and for any cohomology class α in $H^\bullet(BK, \mathbb{R})$, the class $f^*\alpha$ in $H^\bullet(M, \mathbb{R})$ is represented by the differential form $\Phi(\alpha)(F_\nabla)$, where F_∇ is the curvature of any connection on the principal bundle f^*EK . We denote by $\text{Sym}_{\mathbb{Z}}^\bullet(\mathfrak{k})^K$ the image by Φ of $H^\bullet(BK, \mathbb{Z})$.

Let ∇ and ∇^U denote respectively the connections on the K -principal bundles over G/K and G_U/K given by the distribution orthogonal to the fibers (with respect to the Killing metric). These connections (hence their curvature forms) are respectively G and G_U -invariant.

By the preceeding remarks, there is a polynomial $P \in \text{Sym}_{\mathbb{Z}}^{\bullet}(\mathfrak{k})^K$ such that $\frac{d}{\text{Vol}(G_U/H_U)}\omega_{G,H}^U$ and $P(F_{\nabla^U})$ are cohomologous. Since both forms are G_U -invariant, we actually have

$$\frac{d}{\text{Vol}(G_U/H_U)}\omega_{G,H}^U = P(F_{\nabla^U}) .$$

By duality between the symmetric spaces G_U/K and G/K , we then have

$$\frac{d}{\text{Vol}(G_U/H_U)}\omega_{G,H} = (-1)^{\deg P} P(F_{\nabla}) .$$

Let us denote by α the inverse image of P by the Chern–Weil isomorphism Φ .

By Theorem 2, we have

$$\begin{aligned} d \frac{\text{Vol}(\Gamma \backslash G/H)}{\text{Vol}(G_U/H_U)} &= \left| \int_{[\Gamma]} \frac{d}{\text{Vol}(G_U/H_U)} \omega_{G,H} \right| \\ &= \left| \int_{[\Gamma]} P(F_{\nabla}) \right| \\ &= \left| \int_{[\Gamma]} f^* \alpha \right| , \end{aligned}$$

where $f : \Gamma \backslash G/K \rightarrow BK$ is such that the K -principal bundle $\Gamma \backslash G$ over $\Gamma \backslash G/K$ is isomorphic to f^*EK . Since α belongs to $H^{\bullet}(BK, \mathbb{Z})$, we obtain that $\frac{d\text{Vol}(\Gamma \backslash G/H)}{\text{Vol}(G_U/H_U)}$ is an integer. This proves Theorem 5.1. \square

Finally, let us conclude the proof of Theorem 3. Recall that the complex rank of $\text{SO}(n)$ is $\lfloor \frac{n}{2} \rfloor$ and that the complex rank $\text{SL}(n, \mathbb{R})$ is $n-1$. It is then a simple computation to verify that the equality $\mathbf{rk}(H_U) - \mathbf{rk}(L) = \mathbf{rk}(G_U) - \mathbf{rk}(K)$ is satisfied in cases (1) and (2). For case (3), it is a well-known fact that $\mathbf{rk}(G_U) = \mathbf{rk}(K)$ when G_U/K is Hermitian (see [?, Proposition 2.3]). In that case, any G_U -invariant form is a Chern–Weil form. In particular, $\omega_{G,H}^U$ is a Chern–Weil form (which vanishes if $\mathbf{rk}(H_U) - \mathbf{rk}(L) > 0$).

6. THE CASE OF GROUP MANIFOLDS

In this section, we specify the previous results in the case of compact quotients of *group spaces*.

Definition 6.1. A group space is a semi-simple Lie group H provided with the action of $H \times H$ given by

$$(g, h) \cdot x = gxh^{-1}$$

for all $(g, h) \in H \times H$ and all $x \in H$.

The group space H can also be presented as the quotient $H \times H / \Delta(H)$, where $\Delta(H)$ denotes the diagonal embedding of H in $H \times H$.

Group spaces form a large class of pseudo-Riemannian symmetric spaces (the pseudo-Riemannian metric being the Killing metric on H) which is interesting to study for several reasons.

First, given a compact Clifford–Klein form $\Gamma \backslash G / H$ of a reductive homogeneous space and a uniform lattice Λ in H , one can construct the double quotient

$$\Gamma \backslash G / \Lambda ,$$

which is a compact Clifford–Klein form of the group space G . In order to understand all compact Clifford–Klein forms of reductive homogeneous spaces, it is thus enough (in theory) to understand compact quotients of group spaces.

The second motivation for studying group spaces is that, when H has rank one, its compact Clifford–Klein forms are well-understood, thanks to results of Kobayashi [18, 20], Kassel [12], Guéritaud [11], Guichard and Wienhard [10].

Let Γ be a uniform lattice in H and $\rho : \Gamma \rightarrow H$ a homomorphism. We denote by Γ_ρ the graph of ρ , i.e. the subgroup of $H \times H$ defined by

$$\Gamma_\rho = \{(\gamma, \rho(\gamma)), \gamma \in \Gamma\} .$$

The *translation length* of an element $h \in H$ is defined by

$$l(h) = \inf_{x \in H/L} d(x, h \cdot x) ,$$

where d is the distance associated to the H -invariant symmetric Riemannian metric on H/L . We say that the homomorphism ρ is *uniformly contracting* if there exists $\lambda < 1$ such that for any $\gamma \in \Gamma$,

$$l(\rho(\gamma)) \leq \lambda l(\gamma) .$$

Theorem 6.2 (Kobayashi [18], Kassel [12], Guéritaud–Guichard–Kassel–Wienhard [10]). *Let H be a Lie group of rank 1. Then every torsion-free discrete subgroup of $H \times H$ acting properly discontinuously and cocompactly on H is equal to Γ_ρ for some uniform lattice Γ in H and some contracting homomorphism $\rho : \Gamma \rightarrow H$.*

Conversely, Benoist–Kobayashi’s properness criterion [3, 19] implies that such a group Γ_ρ does act properly discontinuously and cocompactly on H .

The purpose of this section is to express the volume of $\Gamma_\rho \backslash H$ when $H = \mathrm{SO}_0(d, 1)$ or $\mathrm{SU}(d, 1)$ in terms of classical invariants associated to the representation ρ .¹ In the case of $\mathrm{SO}_0(d, 1)$, we will recover the main theorem of [38].

In order to do so, we first give a general way to compute the form $\omega_{G, H}$ for any group space $H \times H / \Delta(H)$, knowing the algebra of H -invariant forms on H/L . We thus restrict to the case where $G = H \times H$ acts on $X = H$ by left

¹The case where H is another Lie group of rank 1 (namely $\mathrm{Sp}(d, 1)$ or F_4) is not interesting because the representation ρ must be virtually trivial, according to the super-rigidity theorem of Corlette [9].

and right multiplication. To simplify notations, we denote by ω_H the form $\omega_{H \times H, \Delta(H)}$ constructed in Section 2 and by ω_H^U the corresponding form on the compact dual. The forms ω_H and ω_H^U are respectively a $H \times H$ -invariant form on $H/L \times H/L$ and a $H_U \times H_U$ -invariant form on $H_U/L \times H_U/L$.

Let X be a compact oriented manifold of dimension d . We denote by \vee the homological intersection pairing of X and by \wedge the cohomological product. For $0 \leq k \leq d$, let us fix a basis $(e_1^k, \dots, e_{n_k}^k)$ of the torsion-free part of $H_k(X, \mathbb{Z})$. Let us denote by $(e_1^{k*}, \dots, e_{n_k}^{k*})$ the dual basis for the intersection pairing, i.e. the basis of the torsion-free part of $H_{d-k}(X, \mathbb{Z})$ characterized by

$$e_i^k \vee e_j^{d-k} = \delta_{ij}.$$

Finally, let us denote by $(\alpha_1^k, \dots, \alpha_{n_k}^k)$ and $(\alpha_1^{k*}, \dots, \alpha_{n_k}^{k*})$ the bases of $H^k(X, \mathbb{Q})$ and $H^{d-k}(X, \mathbb{Q})$ satisfying respectively

$$\int_{e_i^k} \alpha_j^k = \delta_{ij}$$

and

$$\int_{e_i^{k*}} \alpha_j^{k*} = \delta_{ij}.$$

Recall that the cohomology ring of $X \times X$ is naturally isomorphic to the tensor product

$$H^\bullet(X, \mathbb{Q}) \otimes H^\bullet(X, \mathbb{Q}).$$

Definition 6.3. We call *Lefschetz cohomology class* on $X \times X$ the cohomology class of degree d defined by

$$\beta_{Lef} = \sum_{k=0}^d (-1)^{d-k} \sum_{i=1}^{n_k} \alpha_i^k \otimes \alpha_i^{k*}.$$

The Lefschetz cohomology class on $H_U/L \times H_U/L$ can be represented by a unique $H_U \times H_U$ -invariant form that we call the *Lefschetz form*. We also call *Lefschetz form* the corresponding $H \times H$ -invariant form on the dual symmetric space $H/L \times H/L$.

The following proposition characterizes the Lefschetz cohomology class and shows in particular that it does not depend on our choice of basis for the homology.

Proposition 6.4. *The Lefschetz cohomology class of X is Poincaré-dual to the diagonal embedding of X in $X \times X$.*

In particular, when integrating the Lefschetz cohomology class on the graph of some map $f : X \rightarrow X$, one recovers the Lefschetz trace formula. Hence our choice of terminology.

Proof. Let Δ_X denote the diagonal embedding of X in $X \times X$. We want to prove that for any $u \in H_d(X \times X, \mathbb{Q})$, the number $\int_u \beta_{Lef}$ equals the

homological intersection number between u and Δ_X . Since

$$H_d(X, \mathbb{Q}) = \bigoplus_{k=0}^d H_k(X, \mathbb{Q}) \otimes H_{d-k}(X, \mathbb{Q}) ,$$

it is enough to prove it for u of the form $e_i^k \otimes e_j^{k*}$, for all $0 \leq k \leq d$ and all $1 \leq i, j \leq n_k$.

By definition of β_{Lef} , we have

$$\int_{e_i^k \otimes e_j^{k*}} \beta_{Lef} = (-1)^{d-k} \delta_{ij} .$$

On the other side, intersections between (cycles representing) $e_i^k \otimes e_j^{k*}$ and Δ_X correspond exactly to intersections between e_i^k and e_j^{k*} . Indeed, e_i^k intersects e_j^{k*} at a point $x \in X$ if and only if $e_i^k \times e_j^{n-k}$ intersects Δ_X at (x, x) . Taking orientations into account, one checks that a positive intersection between e_i^k and e_j^{k*} gives an intersection of sign $(-1)^{d-k}$ between $e_i^k \otimes e_j^{k*}$ and Δ_X . We thus obtain

$$(e_i^k \otimes e_j^{k*}) \vee \Delta_X = (-1)^{d-k} e_i^k \vee e_j^{k*} = (-1)^{d-k} \delta_{ij} .$$

□

By Proposition 3.2, the form $\frac{1}{\text{Vol}(H_U)} \omega_H^U$ on $G_U/K = H_U/L \times H_U/L$ is Poincaré dual to the diagonal embedding of H_U/L . By Propostion 6.4, we thus get:

Corollary 6.5. *The form $\frac{1}{\text{Vol}(H_U)} \omega_H$ is the Lefschetz form on $H/L \times H/L$.*

Let us now apply this corollary to the case where H is $\text{SO}_0(d, 1)$ or $\text{SU}(n, 1)$.

Let $\text{vol}_{\mathbb{H}^d}$ denote the volume form on the hyperbolic space \mathbb{H}^d , which is the symmetric space of $\text{SO}_0(d, 1)$. If Γ is a uniform lattice in $\text{SO}_0(d, 1)$ and $\rho : \Gamma \rightarrow \text{SO}_0(d, 1)$ a homomorphism, we define the *volume* of ρ by

$$\text{Vol}(\rho) = \int_{\mathbb{H}^d/\Gamma} f^* \text{vol}_{\mathbb{H}^d} ,$$

where $f : \mathbb{H}^d \rightarrow \mathbb{H}^d$ is any ρ -equivariant map.

Let ω denote the Kähler form on the complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^d$, which is the symmetric space of $\text{SU}(d, 1)$. We normalize ω so that the corresponding form on the compact dual symmetric space \mathbb{CP}^d is a generator of $H^2(\mathbb{CP}^d, \mathbb{Z})$. If Γ is a uniform lattice in $\text{SU}(d, 1)$ and $\rho : \Gamma \rightarrow \text{SU}(d, 1)$ a homomorphism, we define

$$\tau_k(\rho) = \int_{\Gamma \backslash \mathbb{H}_{\mathbb{C}}^d} f^* \omega^k \wedge \omega^{d-k} ,$$

where $f : \mathbb{H}_{\mathbb{C}}^d \rightarrow \mathbb{H}_{\mathbb{C}}^d$ is any smooth ρ -equivariant map. The number $\tau_1(\rho)$ is often called the *Toledo invariant* of ρ , while $\tau_d(\rho)$ is the *volume* of the representation ρ .

Theorem 6.6.

- If Γ is a uniform lattice in $\mathrm{SO}_0(d, 1)$ and $\rho : \Gamma \rightarrow \mathrm{SO}_0(d, 1)$ a uniformly contracting representation, then

$$\mathbf{Vol}(\Gamma_\rho \backslash \mathrm{SO}_0(d, 1)) = \mathbf{Vol}(\mathrm{SO}(d)) \left| \mathbf{Vol}(\Gamma \backslash \mathbb{H}^d) + (-1)^d \mathbf{Vol}(\rho) \right|.$$

- If Γ is a uniform lattice in $\mathrm{SU}(d, 1)$ and $\rho : \Gamma \rightarrow \mathrm{SU}(d, 1)$ is a uniformly contracting representation, then

$$\mathbf{Vol}(\Gamma_\rho \backslash \mathrm{SU}(d, 1)) = \mathbf{Vol}(\mathrm{SU}(d+1)) \left| \sum_{k=0}^d \tau_k(\rho) \right|.$$

Proof. The compact symmetric space dual to \mathbb{H}^d is \mathbb{S}^d , whose cohomology ring is generated by $\mathbf{1}$ and the fundamental class. We deduce that the Lefschetz form of $\mathbb{H}^d \times \mathbb{H}^d$ is

$$\frac{1}{\mathbf{Vol}(\mathbb{S}^d)} \left(\mathrm{vol}_{\mathbb{H}^d} \otimes \mathbf{1} + (-1)^d \mathbf{1} \otimes \mathrm{vol}_{\mathbb{H}^d} \right).$$

Clearly, Theorem 6.6 is consistent with taking finite index subgroups. By Selberg's lemma, we can thus assume that Γ is torsion-free. Let $f : \mathbb{H}^d \rightarrow \mathbb{H}^d$ be a smooth ρ -equivariant map. Then the graph of f is a Γ_ρ -invariant submanifold of dimension d of $\mathbb{H}^d \times \mathbb{H}^d$ on which Γ_ρ acts freely, properly discontinuously and cocompactly. Let us denote by $\mathrm{Graph}(f)$ its quotient by Γ_ρ :

$$\mathrm{Graph}(f) = \Gamma_\rho \backslash \{(x, f(x)), x \in \mathbb{H}^d\} \subset \Gamma_\rho \backslash \mathbb{H}^d \times \mathbb{H}^d.$$

Then $\mathrm{Graph}(f)$ represents the homology class $[\Gamma_\rho]$ and by Theorem 2, we have

$$\begin{aligned} \mathbf{Vol}(\Gamma_\rho \backslash \mathrm{SO}_0(d, 1)) &= \frac{\mathbf{Vol}(\mathrm{SO}(d+1))}{\mathbf{Vol}(\mathbb{S}^d)} \left| \int_{\mathrm{Graph}(f)} \mathrm{vol}_{\mathbb{H}^d} \otimes \mathbf{1} + (-1)^d \mathbf{1} \otimes \mathrm{vol}_{\mathbb{H}^d} \right| \\ &= \mathbf{Vol}(\mathrm{SO}(d)) \left| \int_{\Gamma \backslash \mathbb{H}^d} \mathrm{vol}_{\mathbb{H}^d} \wedge f^* \mathbf{1} + (-1)^d \mathbf{1} \wedge f^* \mathrm{vol}_{\mathbb{H}^d} \right| \\ &= \mathbf{Vol}(\mathrm{SO}(d)) \left| \mathbf{Vol}(\Gamma \backslash \mathbb{H}^d) + (-1)^d \mathbf{Vol}(\rho) \right|. \end{aligned}$$

Similarly, the integral cohomology ring of \mathbb{CP}^d is generated by the powers of the form symplectic form ω^U . We deduce that the Lefschetz form of $\mathbb{H}_{\mathbb{C}}^d \times \mathbb{H}_{\mathbb{C}}^d$ is

$$\sum_{k=0}^d \omega^k \otimes \omega^{d-k}.$$

Let $f : \mathbb{H}_{\mathbb{C}}^d \rightarrow \mathbb{H}_{\mathbb{C}}^d$ be a smooth ρ -equivariant map and define

$$\mathrm{Graph}(f) = \Gamma_\rho \backslash \{(x, f(x)), x \in \mathbb{H}_{\mathbb{C}}^d\} \subset \Gamma_\rho \backslash \mathbb{H}_{\mathbb{C}}^d \times \mathbb{H}_{\mathbb{C}}^d.$$

As in the $\mathrm{SO}_0(d, 1)$ case, we have

$$\begin{aligned}
\mathrm{Vol}(\Gamma_\rho \backslash \mathrm{SU}(d, 1)) &= \mathrm{Vol}(\mathrm{SU}(d+1)) \left| \int_{\mathrm{Graph}(f)} \sum_{k=0}^d \omega^k \otimes \omega^{d-k} \right| \\
&= \mathrm{Vol}(\mathrm{SU}(d+1)) \left| \sum_{k=0}^d \int_{\Gamma \backslash \mathbb{H}_{\mathbb{C}}^d} \omega^k \wedge f^* \omega^{d-k} \right| \\
&= \mathrm{Vol}(\mathrm{SU}(d+1)) \left| \sum_{k=0}^d \tau_k(\rho) \right|.
\end{aligned}$$

□

7. OBSTRUCTION TO THE EXISTENCE OF COMPACT CLIFFORD–KLEIN FORMS

In this section, we return to the general case of a reductive homogeneous space G/H .

Assume that the form $\omega_{G,H}$ (or equivalently, the form $\omega_{G,H}^U$) vanishes. Then Theorem 2 implies that the volume of a compact quotient of G/H should be 0. Therefore, such a compact quotient simply cannot exist.

As a first application of this obstruction, one obtains a proof of Kobayashi's rank conjecture (Theorem 6), which follows directly from the first point of Theorem 4.1:

Theorem 7.1. *If $\mathrm{rk}(G) - \mathrm{rk}(K) < \mathrm{rk}(H) - \mathrm{rk}(L)$, then G/H does not have compact quotients.*

Unfortunately, this theorem does not provide any new example of homogeneous spaces without compact quotients. Indeed, Morita independently proved in [30] that this theorem is implied by the cohomological obstruction he described in [32].

In this section, we give three other ways of proving that the form $\omega_{G,H}$ vanishes, leading to the proof of Theorem 5.

Theorem 7.2. *For the following pairs (G, H) , the volume form $\omega_{G,H}$ vanishes and G/H does not admit any compact Clifford–Klein form.*

- (1) $G = \mathrm{SO}_0(p, q+r)$, $H = \mathrm{SO}_0(p, q)$, $p, q, r > 0$, p odd;
- (2) $G = \mathrm{SL}(n, \mathbb{R})$, $H = \mathrm{SL}(m, \mathbb{R})$, $1 < m < n$, m even.

Proof of Theorem 7.2. Recall that, by Lemma 2.4, the form $\omega_{G,H}$ at the point $x_0 = K$ is given by

$$(\omega_{G,H})_{x_0} = \int_{K/L} \mathrm{Ad}_u^* \omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp} \, d\omega_{K/L}(u) .$$

In both cases, we exhibit an element $\Omega \in K$ whose action on \mathfrak{g} stabilizes $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ and whose induced action on $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ has determinant -1 . It follows

that

$$\begin{aligned}
\omega_{G,H} &= \int_{K/L} \text{Ad}_{U*} \omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp} \, d\text{vol}_{K/L}(U) \\
&= \int_{K/L} \text{Ad}_{U\Omega*} \omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp} \, d\text{vol}_{K/L}(U) \\
&= \int_{K/L} -\text{Ad}_{U*} \omega_{\mathfrak{k}^\perp \cap \mathfrak{h}^\perp} \, d\text{vol}_{K/L}(U) \\
&= -\omega_{G,H} ,
\end{aligned}$$

hence $\omega_{G,H} = 0$.

For both cases in Theorem 7.2, we now describe $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ as a space of matrices and we give a choice of an element Ω . This element Ω simply multiplies certain coefficients of the matrices in $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ by -1 and we leave to the reader the verification that the induced action on $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ has determinant -1 .

- (1) $G = \text{SO}_0(p, q + r)$, $H = \text{SO}_0(p, q)$, $p, q, r > 0$, p odd:

In this case, $K = \text{SO}(p) \times \text{SO}(q + r)$ and $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ is the space of matrices of the form

$$\left(\begin{array}{c|c|c} 0 & 0 & A^T \\ \hline 0 & & \\ \hline A & 0 & \end{array} \right) ,$$

with $A \in \mathcal{M}_{r,p}(\mathbb{R})$. We take Ω to be the diagonal matrix such that $\Omega_{ii} = -1$ when $i = p + q$ or $p + q + 1$ and $\Omega_{ii} = 1$ otherwise.

- (2) $G = \text{SL}(n, \mathbb{R})$, $H = \text{SL}(m, \mathbb{R})$, m even:

In this case, $K = \text{SO}(n)$ and $\mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ is the space of matrices of the form

$$\left(\begin{array}{c|c} \lambda \text{I}_m & A \\ \hline A^T & B \end{array} \right) ,$$

with $A \in \mathcal{M}_{m,n-m}(\mathbb{R})$, $B \in \text{Sym}_{n-m}(\mathbb{R})$ and $\lambda \in \mathbb{R}$ satisfying $\text{Tr}(B) + m\lambda = 0$. We take Ω to be the diagonal matrix such that $\Omega_{ii} = -1$ when $i = m$ or $m + 1$ and $\Omega_{ii} = 1$ otherwise.

□

We now turn to another way of proving that $\omega_{G,H}$ vanishes. Recall that $\omega_{G,H}$ vanishes if and only if the corresponding form $\omega_{G,H}^U$ on G_U/K vanishes. By Theorem 3.2, this happens whenever $\iota_*[H_U/L]$ vanishes in $H_\bullet(G_U/K, \mathbb{Q})$.

Theorem 7.3. *If G is the complexification of H , then the form $\omega_{G,H}$ vanishes if and only if $H_{\text{even}}^\bullet(H_U/L, \mathbb{Q}) \neq 0$. In particular, for the following pairs (G, H) , the space G/H has no compact Clifford–Klein form:*

- (3) $G = \text{SO}(p+q, \mathbb{C})$, $H = \text{SO}_0(p, q)$, $p, q > 1$ or $p = 1$ and q even;
- (4) $G = \text{SL}(p+q, \mathbb{C})$, $H = \text{SU}(p, q)$, $p, q > 0$;
- (5) $G = \text{Sp}(2(p+q), \mathbb{C})$, $H = \text{Sp}(p, q)$;
- (6) $G = \text{SO}(2n, \mathbb{C})$, $H = \text{SO}^*(2n)$.

Proof. Since G is the complexification of H , we have $H_U = K$. Since G is a complex Lie group, we have $G_U = K \times K$. It follows that G_U/K is the group space K and that $H_U/L = K/L$ is mapped to K by

$$\iota : g \mapsto g \theta(g)^{-1},$$

where θ is the involution of H_U whose fixed point set is L . By Proposition 3.2, $\omega_{G,H}$ does not vanish if and only if $\iota_*[H_U/L]$ does not vanish in $H_\bullet(H_U, \mathbb{Q})$, which happens if and only if the image of ι^* contains a non-zero cohomology class of degree $\dim(H_U/L)$.

By the work of Cartan [7], the cohomology algebra of H_U is generated by bi-invariant forms of odd degree. Moreover, ι^* maps $H^\bullet(H_U, \mathbb{Q})$ surjectively to $H_{\text{odd}}^\bullet(H_U/L, \mathbb{Q})$. Since $H^\bullet(H_U/L, \mathbb{Q}) = H_{\text{odd}}^\bullet(H_U/L, \mathbb{Q}) \otimes H_{\text{even}}^\bullet(H_U/L, \mathbb{Q})$, the image of ι^* contains a form of top degree if and only if $H_{\text{even}}^\bullet(H_U/L, \mathbb{Q}) \equiv 0$.

Let us now prove (3), (4), (5) and (6). For $H = \text{SU}(p, q)$, $\text{Sp}(p, q)$, $\text{SO}^*(2n)$ or $\text{SO}_0(p, q)$ with p or q even, one actually has $\mathbf{rk}(H_U) = \mathbf{rk}(L)$. Therefore the cohomology of H_U/L is concentrated in even degree and the image of the map ι^* is trivial. In particular, it does not contain a non-zero class of top degree.

It remains to treat the case where $H = \text{SO}_0(p, q)$ with p and q odd. Note that H_U/L is the Grassmannian of p -planes in \mathbb{R}^{p+q} . In that case, $\mathbf{rk}(H_U) - \mathbf{rk}(L) = 1$ and $H_{\text{odd}}^\bullet(H_U/L)$ thus has dimension 1. If $H_{\text{even}}^\bullet(H_U/L)$ vanished, then the whole cohomology algebra of H_U/L would be one dimensional. This is well-known to be true if and only if p or q equals 1. \square

Theorem 7.4. *For the following pairs (G, H) , the volume form $\omega_{G,H}$ vanishes and G/H does not admit any compact Clifford–Klein form.*

- (7) $G = \text{SL}(p+q, \mathbb{R})$, $H = \text{SO}_0(p, q)$, $p, q > 1$;
- (8) $G = \text{SL}(p+q, \mathcal{H})$, $H = \text{Sp}(p, q)$, $p, q > 1$.

(Here \mathcal{H} denotes the field of quaternions.)

Proof. Again, we prove that $\iota_*[H_U/L]$ vanishes in $H_\bullet(G_U/K)$, this time by showing that H_U/L is homotopically trivial in G_U/K .

The compact dual to $\text{SL}(p+q, \mathbb{R})$ is $\text{SU}(p+q)$. Let us set $V = \mathbb{R}^p \times \{0\}$ and $W = \{0\} \times \mathbb{R}^q$ in \mathbb{C}^{p+q} . Then we can identify K with $\text{Stab}(V \oplus W) \subset \text{SU}(p+q)$, H_U with $\text{Stab}(V \oplus iW)$, and L with $\text{Stab}(V) \cap \text{Stab}(W)$.

For $t \in [0, 1]$, let g_t be the map in $U(p + q)$ defined by

$$\begin{aligned} g_t(x) &= x \text{ if } x \in V, \\ g_t(x) &= e^{\frac{it\pi}{2}} x \text{ if } x \in W. \end{aligned}$$

The conjugation by g_t preserves L and one can thus define

$$\begin{aligned} \varphi_t : H_U/L &\rightarrow G_U/K \\ hL &\mapsto g_t h g_t^{-1} K. \end{aligned}$$

The conjugation by g_t sends $H_U = \text{Stab}(V \oplus iW)$ to $\text{Stab}(V \oplus ie^{\frac{it\pi}{2}}W)$. In particular, φ_0 is the map $\iota : H_U/L \rightarrow G/K$, and φ_1 sends H_U/L to a point. Therefore the map $\iota : H_U/L \rightarrow G_U/K$ is homotopically trivial, and in particular $\iota_*[H_U/L] = 0$ in $H_\bullet(G_U/K)$.

Case (8) can be treated similarly: set $V = \mathbb{C}^p \times \{0\}$ and $W = \{0\} \times \mathbb{C}^q$ in \mathcal{H}^{p+q} . Then $G_U = \text{Sp}(p + q)$ and one can identify K with $\text{Stab}(V \oplus W)$, H_U with $\text{Stab}(V \oplus jW)$ (where i, j, k denote the three complex structures defining the quaternionic structure of \mathcal{H}), and L with $\text{Stab}(V) \cap \text{Stab}(W)$. One obtains the same contradiction as before by conjugating H_U by the linear transformation g_t that is the identity on V and the multiplication by $e^{\frac{t\pi}{2}j}$ on W . \square

7.1. Relation to earlier works. In the past decades, many different works have been devoted to finding various obstructions to the existence of compact Clifford–Klein forms. Let us detail where Theorems 7.2, 7.3 and 7.4 fit in this literature.

- Case (1) of Theorem 7.2 extends results of Kulkarni [23], Kobayashi–Ono [21] and their recent improvement by Morita [32], where both p and q are assumed to be odd. When specified to $r = 1$, we obtain in particular that $\mathbb{H}^{p,q} = \text{SO}_0(p, q + 1)/\text{SO}_0(p, q)$ does not admit a compact quotient when p is odd. This is an important step toward Kobayashi’s space form conjecture.
- The case of $\text{SL}(n, \mathbb{R})/\text{SL}(m, \mathbb{R})$ has also been extensively studied. It is conjectured that $\text{SL}(n, \mathbb{R})/\text{SL}(m, \mathbb{R})$ never admits a compact quotient for $1 < m < n$ (see for instance [22, Conjecture 3.3.10]). Kobayashi proved that such quotients do not exist for $n < \lceil 3/2m \rceil$ [17] and Labourie, Mozes and Zimmer extended the result to $m \leq n - 3$ with completely different methods ([41], [26], [27]). On the other side, Benoist proved that $\text{SL}(2n + 1, \mathbb{R})/\text{SL}(2n, \mathbb{R})$ does not admit a compact quotient [3]. Case (2) of Theorem 7.2 recovers Benoist’s result² and also implies that $\text{SL}(2n + 2, \mathbb{R})/\text{SL}(2n, \mathbb{R})$ does not admit a compact quotient, which was previously known only for $n = 1$ [35].
- Theorem 7.3 is mostly new. Note that the so-called *Calabi–Markus phenomenon* implies that the symmetric spaces $\text{SL}(n, \mathbb{C})/\text{SL}(n, \mathbb{R})$ and $\text{Sp}(2n, \mathbb{C})/\text{Sp}(2n, \mathbb{R})$ do not admit compact Clifford–Klein forms.

²Benoist’s result is actually stronger: every discrete group acting properly discontinuously on $\text{SL}(2n + 1, \mathbb{R})/\text{SL}(2n, \mathbb{R})$ is virtually Abelian.

Therefore, the only classical Lie groups H for which $H_{\mathbb{C}}/H$ might admit a compact Clifford–Klein form are $\mathrm{SO}(p, 1)$ with p even and $\mathrm{SL}(n, \mathcal{H})$ (where \mathcal{H} denotes the quaternions). Interestingly, the homogeneous space $\mathrm{SO}(8, \mathbb{C})/\mathrm{SO}(7, 1)$ is known to admit compact Clifford–Klein forms (see [22, Corollary 3.3.7]).

- Theorem 7.4 improves a recent result of Morita [32], where p and q are assumed to be odd. It was first proved by Kobayashi when $p = q$ [19] and by Benoist when $p = q + 1$ [3]. More precisely, Benoist proved that every discrete group acting properly discontinuously on $\mathrm{SL}(2p+1)/\mathrm{SO}_0(p, p+1)$ is virtually Abelian (in particular, its action is not cocompact). He also constructed proper actions of a free group of rank 2 as soon as $p \neq q$ or $q + 1$.

The proof of Theorem 7.2 can be adapted to show the vanishing of $\omega_{G,H}$ in many other cases that we did not include because the non-existence of compact Clifford–Klein forms was already known. We can prove for instance that $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(m, \mathbb{R}) \times \mathrm{SL}(n-m, \mathbb{R})$ does not have any compact quotient for $0 < m < n$, n odd (see [3]), that $\mathrm{SO}(n, \mathbb{C})/\mathrm{SO}(m, \mathbb{C}) \times \mathrm{SO}(n-m, \mathbb{C})$ does not have any compact quotient for $1 < m < n-1$, n odd (see [17]), or that $\mathrm{SO}(n, \mathbb{C})/\mathrm{SO}(m, \mathbb{C})$ does not have any compact quotient for $1 < m < n$, m even (see [19, 3]).

7.2. Relation to Yosuke Morita’s work. The first version of this article did not contain Sections 3, 4 and 6. Section 5 stated a theorem of *local rigidity* of the volume and Section 7 contained only a refined version of Theorem 7.2. After our preprint appeared on arXiv, Yosuke Morita posted a preprint where he uses a cohomological obstruction to prove the non-existence of compact quotients of certain reductive homogeneous spaces. In particular, he obtained Theorems 7.2, 7.3 and 7.4. This motivated me to find new ways of proving the vanishing of the form $\omega_{G,H}$ and led me to the compact duality argument and theorems 7.3 and 7.4 which improved significantly this paper.

After discussing with Morita, it seems likely, though not obvious, that our two obstructions are in fact equivalent. We hope to prove this equivalence in a future work.

8. LOCAL FOLIATIONS OF G/H AND GLOBAL FOLIATIONS OF $\Gamma \backslash G/H$

The results of this paper were driven by the idea that compact Clifford–Klein forms $\Gamma \backslash G/H$ should “look like” (K/L) -bundles over a classifying space for Γ . This was suggested by the following theorem:

Theorem 8.1 (Guéritaud–Kassel, [11]). *Let Γ be a discrete torsion-free subgroup of $\mathrm{SO}_0(d, 1) \times \mathrm{SO}_0(d, 1)$ acting properly discontinuously and cocompactly on $\mathrm{SO}_0(d, 1)$ (by left and right multiplication). Then Γ is isomorphic to the fundamental group of a closed hyperbolic d -manifold B , and $\Gamma \backslash \mathrm{SO}_0(d, 1)$ admits a fibration over B with fibers of the form*

$$g\mathrm{SO}(d)h^{-1}, \quad g, h \in \mathrm{SO}_0(d, 1) .$$

More generally, we conjecture the following:

Conjecture. *Let G/H be a reductive homogeneous space (with G and H connected), L a maximal compact subgroup of H and K a maximal compact subgroup of G containing L . Let Γ be a torsion free discrete subgroup of G acting properly discontinuously and cocompactly on G/H . Then there exists a closed manifold B of dimension p such that*

- *the fundamental group of B is isomorphic to Γ ,*
- *the universal cover of B is contractible,*
- *$\Gamma \backslash G/H$ admits a fibration over B with fibers of the form gK/L for some $g \in G$.*

To support this conjecture, we note that the vanishing of the form $\omega_{G,H}$ (which implies the non-existence of compact Clifford–Klein forms) is actually an obstruction to the existence of a *local* fibration by copies of K/L .

Proposition 8.2. *Let G/H be a reductive homogeneous space (with G and H connected), L a maximal compact subgroup of H and K a maximal compact subgroup of G containing L . If the form $\omega_{G,H}$ on G/K vanishes (and in particular for all the pairs (G, H) in Theorem 5), then no non-empty open domain of G/H admits a foliation with leaves of the form gK/L .*

The non-existence of such local foliations in certain homogeneous spaces may be quite surprising. For instance, if $G = \mathrm{SO}_0(2n-1, 2)$ and $H = \mathrm{SO}_0(2n-1, 1)$, then G/H is the *anti-de Sitter space* AdS_{2n} (for which the non-existence of compact Clifford–Klein forms was proven by Kulkarni [23]). In that case, K/L is a timelike geodesic and we obtain the following corollary:

Corollary 8.3. *No open domain of the even dimensional anti-de Sitter space can be foliated by complete timelike geodesics.*

This leads to the following more general question, that may be of independent interest:

Question 8.4. *Let G/H be a reductive homogeneous space, G' a closed subgroup of G and $H' = G' \cap H$. When does G/H admit an open domain with a foliation by leaves of the form gG'/H' ?*

Proof of Proposition 8.2. Assume that there exists a non-empty domain U in $X = G/H$ with a foliation by leaves $(F_v)_{v \in V}$ of the form $g_v K/L$. Since the stabilizer in G of $K/L \subset G/H$ is exactly K , the space of leaves V can be seen as a submanifold of dimension p in G/K . Set $U' = \pi^{-1}(V)$, where π is the projection from G/L to G/K . Then the projection ψ from G/L to G/H induces a diffeomorphism from U' to U . We thus have

$$\int_U \mathrm{vol}_X = \int_{U'} \psi^* \mathrm{vol}_X .$$

On the other hand, by construction of $\omega_{G,H}$, we have

$$\int_{U'} \psi^* \mathrm{vol}_X = \int_V \omega_{G,H} .$$

Since U is non-empty, its volume is non-zero, hence the form $\omega_{G,H}$ cannot vanish. \square

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